Homology by metric currents

Samuele Mongodi*

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1 Introduction

Metric currents are, in a certain sense, a metric analogous of flat currents, therefore are related to the geometry of the space and of their support. In this short note, we try to give some evidence for the previous statement, by showing that the homology which can be defined by means of metric normal currents coincides, on nice enough metric spaces, with the usual singular homology.

The "differential forms" used to define metric currents are, in some sense, projections of the metric space to the euclidean space and the metric currents are particular elements of the appropriate dual; this bears a similarity with the usual definition of cohomology, by means of functionals on maps from the euclidean simplexes to the space.

Our main result is contained in Theorem 8 and Corollary 9, asserting the equivalence of the "metric" homology and the usual one; our strategy of proof is quite straightforward: we use the geometric property of metric currents (expressed by Proposition 1) and a generalization of the cone construction (introduced in [1] and already employed in [5] for a different problem) to verify that the axioms of Eilenberg and Steenrod for homology hold.

This result has been mentioned every now and then in the recent literature on metric currents, always without proof (for instance, in [7]). Given the great interest in the geometry of metric spaces, in the last decades, a detailed proof of such a basic theorem could be helpful to spread further light on the phenomena related to the (Lipschitz) geometry of metric spaces, introducing a concrete possibility of using metric currents as a tool to explore and understand them.

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2 The setting

For the definition of metric currents, we refer to [1] and [4]. We will denote by $\mathscr{D}^k$ the space of metric k-differential forms with locally Lipschitz coefficients and by N_k the space of normal currents with compact support, i.e. currents with finite mass whose boundary has finite mass; we will also denote by d the boundary operator.

We recall the following result about metric currents and their supports (cfr. [4, Proposition 3.3]).

Proposition 1 *Let $T \in \mathscr{D}_m(X)$ and $A \subset X$ be a locally compact set containing* $\mathrm{supp}\,(T)$*. Then there is a unique $T_A \in \mathscr{D}_m(A)$ such that*

$$T_A(f, \pi) = T(\tilde{f}, \tilde{\pi})$$

whenever $(f, \pi) \in \mathscr{D}^m(A), (\tilde{f}, \tilde{\pi}) \in \mathscr{D}^m(X)$ and $\tilde{f}|_A = f$, $\tilde{\pi}_i|_A = \pi_i$. Moreover, $\mathrm{supp}\,(T) = \mathrm{supp}\,(T_A)$.

We proceed now to define the homology complex. Given metric space X, we can consider the chain complex

$$\ldots \longrightarrow N_k(X) \xrightarrow{d} N_{k-1}(X) \longrightarrow \ldots \longrightarrow N_1(X) \xrightarrow{d} N_0(X) \longrightarrow 0$$

where $N_k(X)$ is the space of normal metric currents with compact support, and the associated homology

$$H_k(X) = \frac{\mathrm{Ker}\{d : N_k(X) \to N_{k-1}(X)\}}{\mathrm{Im}\{d : N_{k+1}(X) \to N_k(X)\}}$$

Obviously, if $f : X \to Y$ is a Lipschitz map, we obtain the pushforward operator $f_\sharp : N_k(X) \to N_k(Y)$ for every k and, since $f_\sharp$ and d commute, we have an induced operator

$$H(f) : H_k(X) \to H_k(Y)$$

such that $H(\mathrm{Id}) = \mathrm{Id}$ and $H(f \circ g) = H(f) \circ H(g)$. In other words, H is a covariant functor from the category of metric spaces with Lipschitz functions to the category of abelian groups. In what follows we will write f_* instead of $H(f)$.

Moreover, if A is closed subset of X,we define $N_k(X, A)$ setting

$$N_k(X, A) = N_k(X)/N_k(A).$$

3 The axioms of Eilenberg and Steenrod

Since $d : N_k(X) \to N_{k-1}(X)$ sends $N_k(A)$ in $N_{k-1}(A)$ we can consider the relative homology groups $H_k(X, A)$ and we have the long exact sequence of the pair, in the same way of singular homology

$$\dots H_k(A) \to H_k(X) \to H_k(X, A) \stackrel{d'}{\to} H_{k-1}(A) \to H_{k-1}(X) \to \dots$$

where d' is an homomorphism of degree -1.

Proposition 2 *Let $\{U, V\}$ be an open covering of X, let i_U, i_V be the inclusions of $U \cap V$ in U and V respectively and let j_U, j_V be the inclusions of U and V respectively in X. Then the short sequence of chain complexes*

$$0 \to N_*(\bar{U} \cap \bar{V}) \stackrel{(i_U)_* \oplus (i_V)_*}{\longrightarrow} N_*(\bar{U}) \oplus N_*(\bar{V}) \stackrel{(j_U)_* - (j_V)_*}{\longrightarrow} N_*(X) \to 0$$

is exact.

Proof: Given $T \in N_k(U \cap V)$ with $(i_U)_\sharp(T) = 0$, for every form $(f, \pi) \in \mathscr{D}^k(\bar{U})$ we have

$$T(f|_{U\cap V}, \pi|_{U\cap V}) = 0$$

so $T(g, \eta) = 0$ for every $(g, \eta) \in \mathscr{D}^k(\bar{U} \cap \bar{V})$, that is $T = 0$.

Moreover, if $(j_U)_\sharp(T) = (j_V)_\sharp(S)$, with $T \in N_k(\bar{U})$ and $S \in N_k(\bar{V})$, then $\operatorname{supp}((j_U)_\sharp(T)) = \operatorname{supp}((j_V)_\sharp(S)) \subseteq \bar{U} \cap \bar{V}$; this means that $T = (i_U)_\sharp R$ and $S = (i_V)_\sharp R$ with $R \in N_k(\bar{U} \cap \bar{V})$.

Finally, given $T \in N_k(X)$, we can consider a partition of unity subordinated to the covering $\{U, V\}$, $\{\phi_U, \phi_V\}$. The current $T \llcorner \phi_U$ has support contained in U, therefore, by Proposition 1, there is $S_1 \in N_k(\bar{U})$ such that $T \llcorner \phi_U = (j_U)_\sharp S_1$; similarly, there is $S_2 \in N_k(\bar{V})$ such that $-T \llcorner \phi_V = (j_V)_\sharp S_2$. So, we have that $T = (j_U)_\sharp S_1 - (j_V)_\sharp S_2$ and the exactness of the sequence follows. □

By employing the usual techniques of homological algebra and Proposition 2, we can now prove the Mayer-Vietoris sequence theorem for the homology of normal currents.

Proposition 3 *Given a closed subset A of X and an open set U such that $\bar{U}$ is contained in the interior of A, we have that the inclusion map $(X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism in homology.*

Proof: The result follows from the exactness of Mayer-Vietoris sequence, in the same way as in singular homology. □

Proposition 4 *Two locally Lipschitz-homotopic locally Lipschitz maps*

$$f \sim g : X \to Y$$

induce the same homomorphism in homology

Proof: Let $H : X \times [0,1] \to Y$ be the locally Lipschitz homotopy between f and g and let us define the operator

$$K : N_k(X) \to N_{k+1}(Y)$$

by the following formula

$$K(T)(f, \pi_1, \ldots, \pi_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \int_0^1 T\left(f \circ H \cdot \frac{\partial \pi_i \circ H}{\partial t}, \ldots, \hat{\pi_i}, \ldots\right)$$

Arguing like in [1, Proposition 10.2] (see also [5] for another generalization of the same Proposition), we see that if $T \in N_k(X)$, $K(T)$ is also in $N_{k+1}(Y)$ and the following holds

$$d(K(T)) = -K((dT)) + g_\sharp T - f_\sharp T.$$

Consequently, if $dT = 0$, we see that $g_\sharp T - f_\sharp T$ is in the image of $d : N_{k+1}(Y) \to N_k(Y)$, that is $f_* = g_*$ as applications between $H_*(X)$ and $H_*(Y)$. □

Proposition 5 *If X is a metric space with only one point, we have*

$$H_*(X) = \begin{cases} \mathbb{K} & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Proof: The thesis is obvious, as $M_0(X) = \{\alpha\delta_x \mid \alpha \in \mathbb{K}\} \cong \mathbb{K}$ and $M_j(X) = \{0\}$ for $j > 0$. □

Theorem 6 *The functor H_* defines a homology theory.*

Proof: We refer to [2,6] for a complete discussion of the Eilenberg-Steenrod axioms and their variations; we content ourselves with noting that H_* is obviously a functor and that, by the results we proved in the previous pages, it fulfills the homotopy axiom, the excision axiom, the dimension axiom and the exactness of the long sequence of the pair.

Moreover, it is obvious that N_* is additive on the disjoint union of spaces and that d preserves such structure, hence also H_* is additive. □

4 Uniqueness of homology

Let Met_2 be the category whose objects are pairs (X, A) where X is a complete metric space and A is a closed subset in X and whose morphisms are continuous functions. Similarly, let Met_{2L} be the category whose objects are

the same of Met_2 and whose morphisms are locally Lipschitz maps; obviously, if (X, A) is an object in Met_2 (or Met_{2L}), also $(X, \emptyset)$ and $(A, \emptyset)$ are, and all the inclusions between them are morphisms of the category.

Let us denote by Met_2' (resp. Met_{2L}') the category whose objects are the same of Met_2 and whose morphisms are equivalence classes of morphisms of Met_2 (resp. Met_{2L}) with respect to the relation of being homotopic (resp. being homotopic by a locally Lipschitz homotopy).

We will denote by $\mathrm{Mor}_{\mathcal{C}}(O_1, O_2)$ the class of morphisms from O_1 to O_2 in the category $\mathcal{C}$.

Lemma 7 *If (X, A) and (Y, B) are objects of Met_2', then they are also objects of Met_{2L}'. Moreover, if (X, A) and (Y, B) are CW-pairs, then*

$$\mathrm{Mor}_{\mathsf{Met}_2'}((X, A), (Y, B)) = \mathrm{Mor}_{\mathsf{Met}_{2L}'}((X, A), (Y, B))$$

Proof: The first assertion is obvious, by the definition of the categories Met_2' and Met_{2L}'. If (X, A) and (Y, B) are CW-pairs, X and Y are locally finite CW-complexes, as they are metrizable; given a continuous map of pairs $f : (X, A) \to (Y, B)$, by the cellular approximation theorem, we can find a cellular map $f' : (X, A) \to (Y, B)$.

We can assume that the characteristic maps are Lipschitz, up to homotopy, therefore, given a cell e of X, we can consider the Lipschitz map $j : e \to X$ and compose it with f', obtaining $f' \circ j : e \to Y$. If $f' \circ j$ is Lipschitz on the boundary of e, it is easy to pick, in the same homotopy class of $f' \circ j$, a Lipschitz map $g_j : e \to Y$ which coincides with $f' \circ j$ on the boundary of e.

Therefore, we can inductively construct a locally Lipschitz map $g : (X, A) \to (Y, B)$ which is homotopic to f', hence to f. Moreover, if f' was already locally Lipschitz on a sub complex $C \subset X$, we can take g to agree with f on C.

On the other side, if $f_1,\ f_2 : (X, A) \to (Y, B)$ are locally Lipschitz maps, with a continuous homotopy K between them (hence representing the same morphism in Met_2'), we can apply the preceding reasoning to the map

$$K : (X, A) \times (I, 0, 1) \to (Y, B) \ .$$

As $f_1,\ f_2$ are already locally Lipschitz, we can construct a map K', homotopic to K, locally Lipschitz, and such that $K'(x, 0) = f_1(x)$ and $K'(x, 1) = f_2(x)$ for every $x \in X$; therefore, f_1 and f_2 represent the same morphisms also in Met_{2L}'. $\square$

Lemma 7 tells us that there is a natural isomorphism between Met_2' and Met_{2L}'.

We denote by H_*^L the Lipschitz simplicial homology functor (with real coefficients) on Met_2', i.e. the homology (with real coefficients) obtained by

Lipschitz simplicial chains. By [8, Proposition 1.3], on a locally Lipschitz contractible space, H_*^L coincides with the usual simplicial homology; a CW-complex which is also a complete metric space is therefore locally finite, hence it is locally Lipschitz contractible.

Theorem 8 *There exists a natural transformation from the Lipschitz simplicial homology (with real coefficients) H_*^L to H_*, which induces an isomorphism between them on every CW-pair in Met_2.*

Proof: We define a transformation from the chain complex of Lipschitz chains to normal currents with compact support.

Let $\sigma : \Delta_k \to X$ be a Lipschitz k-simplex, let $[e_k]$ be the usual integration current on e_k; $\sigma(\Delta_k)$ is compact in X, hence the k-current $T_\sigma = \sigma_\sharp[e_k]$ has compact support. By the properties of metric currents, T_σ is a normal current; moreover, $[e_n]$ is also a classical current, hence (see for instance [3]) $d[e_n] = [\partial e_n]$, where ∂e_n is the boundary of e_n in the sense of simplicial chains.

Therefore, $dT_\sigma = T_{d\sigma}$. This means that T induces a natural transformation T_* between homologies.

Such a transformation induces an isomorphism between $H_*(X)$ and $H_*^L(X)$ when $X = \{x\}$. Therefore, by the classical results on homology (see [6, Chapter 7]), the two homology theories are isomorphic, when restricted to CW-pairs in Met'_{2L}, hence in Met'_2, hence in Met_2. □

The following corollary is an obvious consequence of the previous theorem and of the result on locally Lipschitz contractible spaces mentioned above.

Corollary 9 *The homology H_* defined in this note coincides with the usual simplicial homology on CW-pairs in Met_2.*

We conclude highlighting some generalizations of the result we obtained, the first one quite straightforward, the second one involving quite a bit of technical details.

Remark 1 *The same argument can be repeated verbatim for the integral homology, employing the space $I_k(X)$ of integral currents and noticing that, by the Boundary Rectifiability Theorem [1, Theorem 8.6], $dI_k(X) \subseteq I_{k-1}(X)$.*

Remark 2 *One could avoid considering just complete metric spaces by modifying the definition of a metric current: if one adds the requirement for the mass of a current to be a tight measure, then one can dispense with completeness, at least as long as the result we employed in this note are concerned. This allows us to repeat the same proof and obtain the same results for non-complete metric spaces.*

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